Integer Programming - Solution *Methods* - Branch and Bound

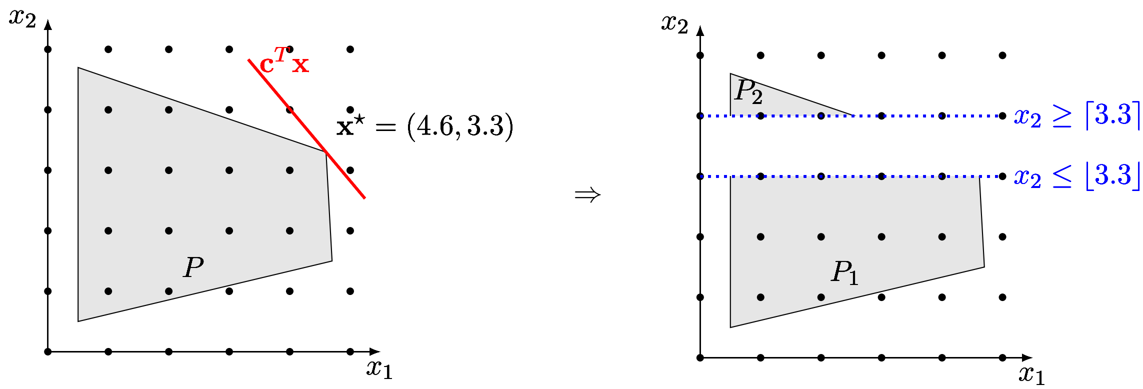
Source: `http://co-at-work.zib.de/files/Gurobi_MIP.pdf`

Problem:

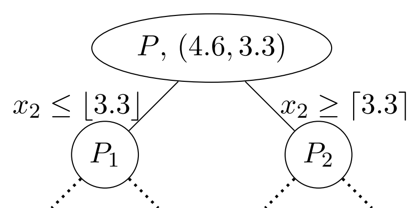
$$(IP)\begin{cases}\text{maximize} & \mathbf{c}^T\mathbf{x}\\ \text{subject to} & A\mathbf{x}\le\mathbf{b},\end{cases}$$

where $\mathbf{c}\in\mathbb{Z}^n, \mathbf{b}\in\mathbb{Z}^m, A\in\mathbb{Z}^{m\times n}$, and $\mathbf{x}\in\mathbb{Z}^n$.

Suppose we try to relax the problem and solve it as a linear programming problem. The set of feasible solutions is P. Suppose that the optimum is $\mathbf{x}^\star=(4.6,3.3)$. We know x_2 cannot be 3.3. So we create two new instances, where we add constraints $x_2\ge\lceil 3.3\rceil$ and $x_2\le\lfloor 3.3\rfloor$. Variable x_2 is a *branch variable*. We solve both instances and better of the solutions is the solution to the original problem.



The same process repeats with P_1 and P_2. Result is a *big* branch and bound tree T.



Branch and (no Bound) outline

1. Let $P=\{\mathbf{x}: A\mathbf{x}\le\mathbf{b}\}$
2. Build tree T with one node P (and mark it unexplored)
3. while T has unexplored node X
4. $\quad\quad$ $\mathbf{x}^\star :=$ optimum for LP relaxation of X; mark X explored
5. $\quad\quad$ If $\mathbf{x}^\star_i\notin\mathbb{Z}$ for some i
6. $\quad\quad\quad\quad$ $X_1 := X\cap\{\mathbf{x}:\mathbf{x}_i\le\lfloor\mathbf{x}^\star_i\rfloor\}$
7. $\quad\quad\quad\quad$ $X_2 := X\cap\{\mathbf{x}:\mathbf{x}_i\ge\lceil\mathbf{x}^\star_i\rceil\}$
8. $\quad\quad\quad\quad$ Add X_1 and X_2 to T as unexplored nodes
9. Return maximum of integer solutions in T.

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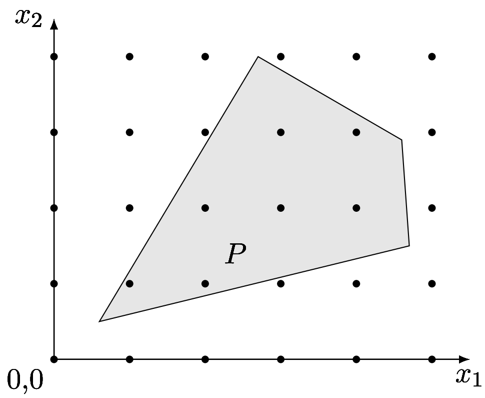
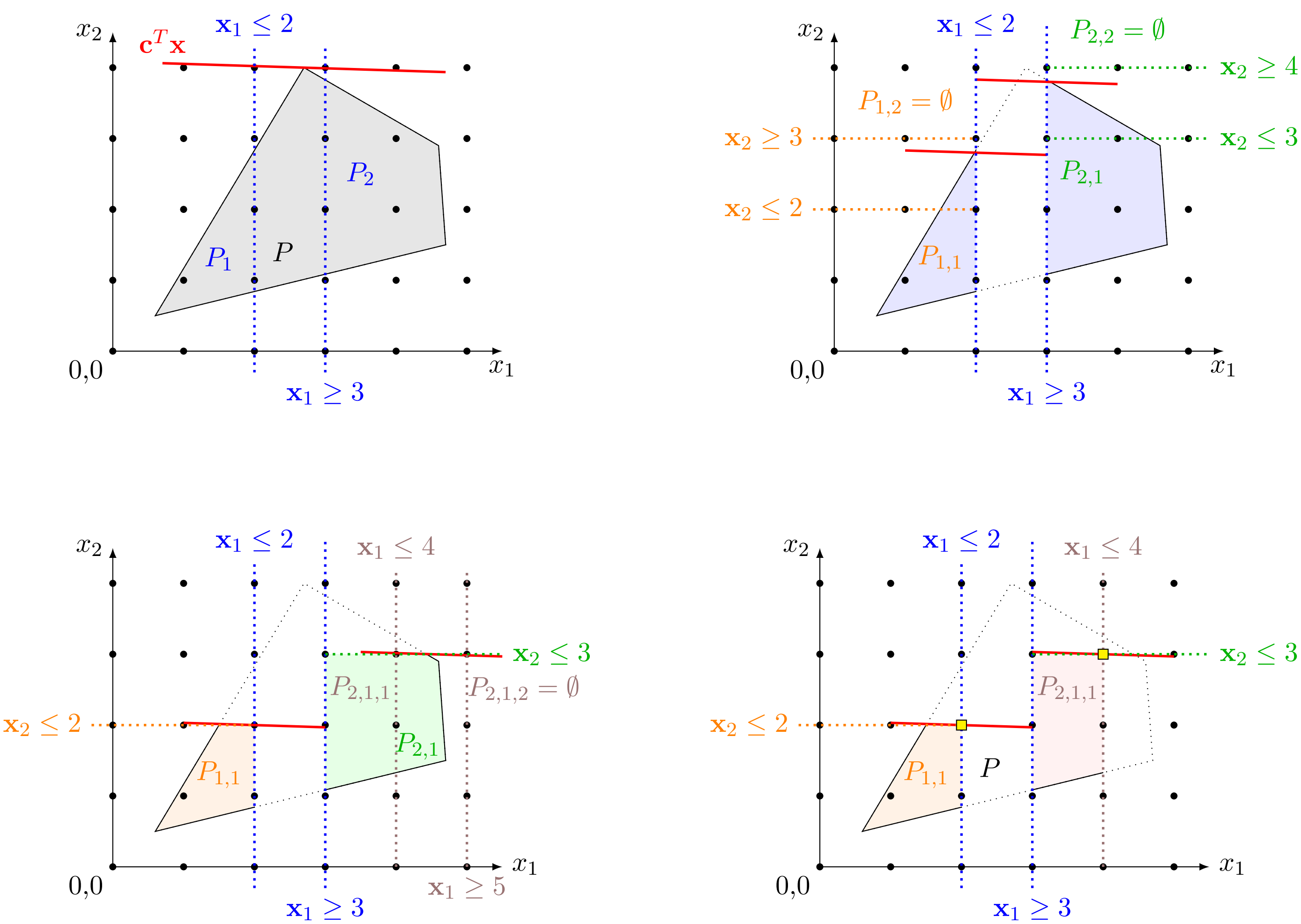
1: Consider problem

$$(IP)\begin{cases}\text{maximize} & 100x_2 + x_1\\ \text{subject to} & (x_1, x_2) \in P,\end{cases}$$

where P is depicted below. Solve (IP) using Brand and Bound. Create branch and bound tree T.



Solution: Here is the sequence of cuttings.



And this is the resulting tree T.

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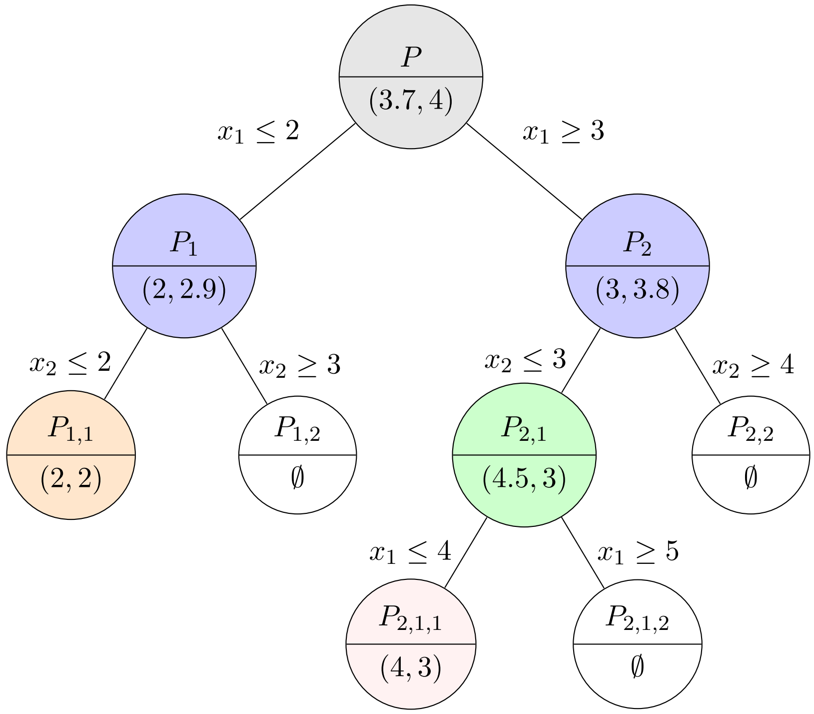

Notice that the leaves have either integer solution or are empty. Also notice that there is more than just one branching on x_1. And the optimum solution is $(4, 3)$, value 304.

2: Will branch and bound ALWAYS find an optimal solution if one exists?

Solution: Yes, this EVENTUALLY gets the right answer.

3: Is there a *good* bound on the size of the tree?

Solution: No - the tree may explode. It may have exponential size.

4: Is it possible to identify nodes in T that will not contain the optimal solution?

Solution: Sometimes. See the example above. Consider we computed node $P_{2,1,1}$ and get an integer solution of value 304. This tells us that the optimum integral solution has value at least 304. Now we look at node P_1 - it gives solution with value 292. In the whole subtree under P_1, all integer solutions in the subtree rooted at P_1 will have value at most 292. Hence no need to solve under P_1. **That is why the method is branch and bound** Note: good idea to try to round and get some integers solutions - helps cut the tree. This is the bound part of the name.

5: What are (dis)advantages of processing nodes deep in the search tree vs nodes close to the root?

Solution: Deep is more likely to give integer solution. But more likely to be eliminated later by some better solution. No clear winner.

6: Which if a solution in a node has more non-integer coordinates, which variable to branch on first?

Solution: Depends on problem - branch on important first. Example - decide if building factory at all before deciding how many production lines it should have.

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